(NASA-TM-82102) PRINCIPAL NORMAL CURVATURE OF SURFACES (NASA) 28 p HC A03/MF A01 CSCL 12A
N81-24801
Unclas
G3/64 25778

NASA

Technical Memorandum 82102

Principal Normal Curvature of Surfaces

R. F. Schmidt

MARCH 1981

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

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ABSTRACT

This document develops certain principal normal curvatures of differential geometry for use in curvature matrices associated with the asymptotic solution of electromagnetic diffraction problems. The effort is directed toward microwave antenna simulations and high speed digital computer analysis of radiometric instruments used to obtain soil moisture, sea state, salinity and temperature data. It is shown that the methods used to develop the principal normal curvatures for paraboloid, hyperboloid, ellipsoid, sphere, and cone can be applied to other radiometer geometries, such as the parabolic torus, even though the surface parameterizations are different. It is concluded that deployable offset geometries, distorted by rotational forces and solar loads may be analyzed by similar means given a suitable surface description.

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ILLUSTRATIONS

GLOSSARY OF NOTATION

I, II, III	Symbols for the three fundamental forms
$\bar{x} = (x, y, z) = \bar{\rho}$	Surface coordinates
$\bar{N} = \bar{n}$	Surface normals
(u, v), (σ, ζ), (x, ξ)	Parameters of surfaces
E, F, G	Vector (dot) products associated with (I)
e, f, g	Vector (box) products associated with (II)
k_1, k_2	Principal normal curvatures of surfaces
α	Azimuthal angle about ($\bar{N}$) in Euler's theorem
k	Generic normal curvature of a surface
$\bar{x}_u$, $\bar{x}_v$	First partial derivatives of ($\bar{x}$)
$\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$	Second partial derivatives of ($\bar{x}$)
F	Focal length of a paraboloid
z_1	Generic z-axis displacement of surface
σ	Radial variable of surfaces
ζ	Angle variable of surfaces
a, c	Hyperboloid or ellipsoid parameters (in context)
$\bar{N}_u$, $\bar{N}_v$	First partials of the normal ($\bar{N}$)
c	Cone parameter (in context)
x	Linear variable of parabolic torus
ξ	Angle variable of parabolic torus
a	Parabolic torus parameter
k_m	Mean curvature

k_g	Gaussian curvature
Q	Curvature matrix
O	Order
ψ	Angle between basis vectors and principal directions
δ	Wave or ray parameter
R_1, R_2	Principal radii of curvature
Γ^k_{ij}	Christoffel symbols of the second kind (functions of E, F, G)
β^i_j	Coefficients (functions of E, F, G, e, f, g)
τ	Torsion
K	Curvature
k_0	Normal curvature of a surface (in context, eqn. 133 only)
k	Curvature of a curve (in context, eqn. 133 only)
ϑ	Angle between principal normal to curve or a surface
γ	Surface

PRINCIPAL NORMALS OF SURFACES

INTRODUCTION

The analytic exploration of electromagnetic scattering from surfaces requires some facility with differential geometry. Uslenghi (Ref. 2), James (Ref. 9), and Collin and Zucker (Ref. 10) associate principal normal curvature of surfaces with the asymptotic solution of electromagnetic edge diffraction at high wave-numbers. Elementary knowledge of curves and surfaces is necessary for the application of ray techniques such as the geometric theory of diffraction. In particular, precise distinctions must be made regarding the curvature of structural members, such as the supporting ribs of deployable offset antennas, and the normal curvature of the surfaces formed by tensioned mesh materials. Analysis of microwave antennas for the radiometric determination of sea state, wind speed, soil moisture, sea temperature, salinity, and snow maps by means of detailed electromagnetic simulation and high speed digital computers relies heavily on rigorous geometric formulations. It is important, therefore, to develop the tangents, normals, differential areas and other information for the well-known surfaces of revolution encountered in the classic literature and also to extend these techniques to the more sophisticated antenna configurations being considered at Goddard Space Flight Center.

In 1968 a physical optics program was initiated at GSFC at which time a useful (σ,ζ) parameterization of the "conics" was employed, Ref. 1, Appendix A. Although the topic of curvature was introduced in that document, the treatment was less than general. Ref. 1, Appendix B. The development of principal normal curvature in the present document may be regarded as an extension of the earlier effort in the same (σ,ζ) parameterization. For convenience the relevant equations for the conics are reproduced here as Appendix A. Reference 3 reviews the conic sections which lead to generating arcs for surfaces with axial symmetry.

The three fundamental forms of differential geometry are usually written as

$$I = d\bar{x} \cdot d\bar{x} \quad (1)$$

$$II = -dx \cdot dN \quad (2)$$

$$III = dN \cdot d\bar{N} \quad (3)$$

where

$$dx = \bar{x}_u\, du + x_v\, dv \quad (4)$$

and

$$dN = \bar{N}_u\, du + \bar{N}_v dv \quad (5)$$

In this notation the unit surface normal ($\bar{N}$) and the surface coordinate vector ($\bar{x}$) are both functions of the parameters (u,v), Ref. 4, p. 103. Ref. 5, Chapter IV.

Development of the theory of surfaces in the classical literature leads to the identification of

$$E = x_u \cdot \bar{x}_u, \quad F = x_u \cdot x_v, \quad G = x_v \cdot x_v \quad (6)$$

and

$$e = \frac{x_{uu}\, x_u\, x_v}{(EG - F^2)^{1/2}} = \bar{x}_{uu} \cdot \bar{N} \quad (7)$$

$$f = \frac{x_{uv}\, x_u\, x_v}{(EG - F^2)^{1/2}} = \bar{x}_{uv} \cdot \bar{N} \quad (8)$$

$$g = \frac{x_{vv}\, x_u\, x_v}{(EG - F^2)^{1/2}} = \bar{x}_{vv} \cdot \bar{N} \quad (9)$$

It can be seen that the following correspondences exist between Appendix A of the present document and the literature:

$$x \equiv \bar{\rho}, \quad x_u \equiv \rho_\sigma, \quad \bar{x}_v \equiv \bar{\rho}_\zeta \quad (10)$$

$$N \equiv \bar{n}, \quad \bar{N}_u \equiv \bar{n}_\sigma, \quad N_v \equiv \bar{n}_\zeta . \quad (11)$$

Ref. 4, p. 58, p. 75.

In the reference above, the expressions for principal normal curvature are given as

$$k_1 = \frac{e}{E}, \quad k_2 = \frac{g}{G} \quad (12)$$

for $dv = 0$ and $du = 0$, respectively. Since the curvatures (k_1) and (k_2) represent extrema, the normal curvature in an arbitrary direction may be written as

$$k = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha , \tag{13}$$

which is known as Euler's theorem, or as

$$k = \frac{II}{I} = \frac{-d\bar{x} \cdot d\bar{N}}{d\bar{x} \cdot d\bar{x}} . \tag{14}$$

Ref. 4, p. 80, p. 81.

A means of verifying the results obtained for the principal normal curvatures of surfaces is available via Rodrigues' formula:

$$d\bar{N} = -k \, d\bar{x} \tag{15}$$

where (k) is the normal curvature in the direction ($d\bar{x}$) of the line of curvature. A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature. That is, equation (14) holds when (k) is given the value associated with the selected principal direction. Ref. 6, p. 186; Ref. 4, p. 94; Ref. 7, p. 211; Ref. 5, p. 97. Then either ($dv = 0$) or ($du = 0$), when a result for (k_1) or (k_2) is verified, since Rodrigues' formula characterizes the principal directions.

The principal normal curvatures for the axially symmetric surfaces whose generating arcs were obtained by conic section are now developed in detail, and verified via equation (15) using a (σ,ζ) parameterization. Subsequently the normal curvatures of an axially symmetric surface described by an (x,ξ) parameterization are developed and verified. Both parameterizations employ one linear (σ or x) variable, and one angle variable (ζ or ξ).

$$\left.\begin{array}{l}\text{paraboloid}\\ \text{hyperboloid}\\ \text{ellipsoid}\\ \text{sphere}\\ \text{cone}\end{array}\right\} (\sigma,\zeta) = (u,v)$$

$$\text{parabolic locus} \left.\right\} (x,\xi) = (u,v)$$

A right-handed Cartesian coordinate system may be assumed throughout. The forms (E, F, G, e, f, g) are written out as they are fundamental for the present development of (k_1) and (k_2) as well as other developments (ds, etc.).

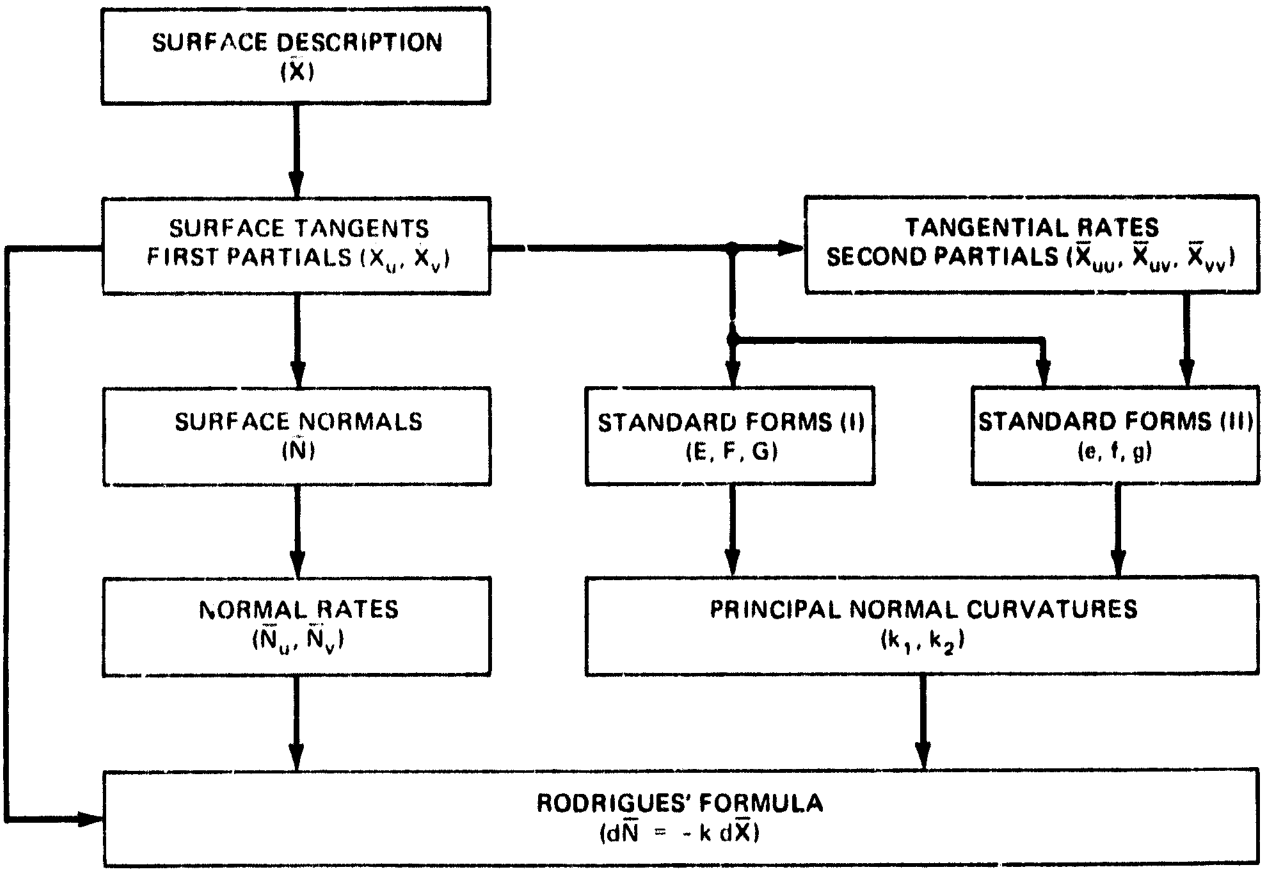

Figure 1. Flow Chart Leading to Rodrigues' Formula

PARABOLOID

$$\bar{x} = \left(\sigma \sin \zeta,\ -\sigma \cos \zeta,\ \frac{\sigma^2}{4F} + z_1\right) \quad \text{surface description} \quad (16)$$

$$\bar{x}_u = (\sin \zeta,\ -\cos \zeta,\ \sigma/2F) \quad \text{tangent to surface} \quad (17)$$

$$\bar{x}_v = (\sigma \cos \zeta,\ \sigma \sin \zeta,\ 0) \quad \text{tangent to surface} \quad (18)$$

$$\bar{x}_{uu} = (0,\ 0,\ 1/2F) \quad \text{tangential rate} \quad (19)$$

$$\bar{x}_{uv} = (\cos \zeta,\ \sin \zeta,\ 0) \quad \text{tangential rate} \quad (20)$$

$$\bar{x}_{vv} = (-\sigma \sin \zeta,\ \sigma \cos \zeta,\ 0) \quad \text{tangential rate} \quad (21)$$

$$E = \bar{x}_u \cdot \bar{x}_u = 1 + \sigma^2/4F^2 \quad (22)$$

$$F = \bar{x}_u \cdot \bar{x}_v = 0 \quad (\perp \text{ systems}) \quad \text{standard forms of (I)} \quad (23)$$

$$G = \bar{x}_v \cdot \bar{x}_v = \sigma^2 \quad (24)$$

$$e = \frac{\bar{x}_{uu}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{1}{(4F^2 + \sigma^2)^{1/2}} \quad (25)$$

$$f = \frac{\bar{x}_{uv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = 0 \quad (\perp \text{ systems}) \quad \text{standard forms of (II)} \quad (26)$$

$$g = \frac{\bar{x}_{vv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{\sigma^2}{(4F^2 + \sigma^2)^{1/2}} \quad (27)$$

$$k_1 = \frac{e}{E} = \frac{4F^2}{(4F^2 + \sigma^2)^{3/2}} \quad \text{principal normal curvature} \quad (28)$$

$$k_2 = \frac{g}{G} = \frac{1}{(4F^2 + \sigma^2)^{1/2}} \quad \text{principal normal curvature} \quad (29)$$

Rodrigues' formula for $(k_1) \rightarrow \bar{N}_u \, du = -k_1 \, \bar{x}_u \, dv$ (30)

$$(k_2) \rightarrow \bar{N}_v \, dv = -k_2 \, \bar{x}_v \, dv \qquad (31)$$

$$\boxed{\bar{N} = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{-\sigma \sin \zeta \, \hat{i} + \sigma \cos \zeta \, \hat{j} + 2F \, \hat{k}}{(\sigma^2 + 4F^2)^{1/2}}} \quad \text{surface normal} \qquad (32)$$

$$\bar{N}_u = \left[\frac{\sigma^2 \sin \zeta}{(\sigma^2 + 4F^2)^{1/2}} - \frac{\sin \zeta}{(\sigma^2 + 4F^2)^{1/2}}\right] \hat{i} \quad \text{normal rate}$$

$$+ \left[\frac{-\sigma^2 \cos \zeta}{(\sigma^2 + 4F^2)^{3/2}} + \frac{\cos \zeta}{(\sigma^2 + 4F^2)^{1/2}}\right] \hat{j}$$

$$+ \left[\frac{-2F\sigma}{(\sigma^2 + 4F^2)^{3/2}}\right] \hat{k} \qquad (33)$$

$$N_v = \frac{(-\sigma \cos \zeta) \, \hat{i} - (\sigma \sin \zeta) \, \hat{j}}{(\sigma^2 + 4F^2)^{1/2}} \quad \text{normal rate} \qquad (34)$$

HYPERBOLOID

$\bar{x} = [\sigma \sin \zeta, -\sigma \cos \zeta, c(1 + \sigma^2/a^2)^{1/2} + z_1]$	surface description	(35)
$\bar{x}_u = [\sin \zeta, -\cos \zeta, (c\sigma/a^2)(1 + \sigma^2/a^2)^{-1/2}]$	tangent to surface	(36)
$\bar{x}_v = [\sigma \cos \zeta, \sigma \sin \zeta, 0]$	tangent to surface	(37)
$\bar{x}_{uu} = [0, 0, (-c\sigma^2/a^4)(1 + \sigma^2/a^2)^{-3/2} + (c/a^2)(1 + \sigma^2/a^2)^{-1/2}]$	tangential rate	(38)
$\bar{x}_{uv} = [\cos \zeta, \sin \zeta, 0]$	tangential rate	(39)
$\bar{x}_{vv} = [-\sigma \sin \zeta, \sigma \cos \zeta, 0]$	tangential rate	(40)

$$E = \bar{x}_u \cdot \bar{x}_u = 1 + \frac{\sigma^2 c^2}{a^2(a^2 + \sigma^2)} \qquad (41)$$

$F = \bar{x}_u \cdot \bar{x}_v = 0$ ($\perp$ systems) standard forms of (I) (42)

$$G = \bar{x}_v \cdot \bar{x}_v = \sigma^2 \qquad (43)$$

$$e = \frac{\bar{x}_{uu}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{ca^2}{(a^2+\sigma^2)\,[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} \tag{44}$$

$$f = \frac{\bar{x}_{uv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = 0 \qquad (\perp \text{ systems}) \tag{45}$$

standard forms of (II)

$$g = \frac{\bar{x}_{vv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{c\sigma^2}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} \tag{46}$$

$$\boxed{k_1 = \frac{e}{E} = \frac{ca^4}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{3/2}}} \tag{47}$$

principal normal curvature

$$\boxed{k_2 = \frac{g}{G} = \frac{c}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}}} \tag{48}$$

principal normal curvature

Rodrigues' formula for $(k_1) \rightarrow \bar{N}_u\,du = -k_1\,\bar{x}_u\,du$ (30)

$$(k_2) \rightarrow \bar{N}_v\,dv = -k_2\,\bar{x}_v\,dv \tag{31}$$

$$\boxed{\bar{N} = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{-c\sigma \sin\zeta\,\hat{i} + c\sigma\cos\zeta\,\hat{j} + a(a^2+\sigma^2)^{1/2}\,\hat{k}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}}} \tag{49}$$

surface normal

$$\begin{aligned}
\bar{N}_u = {} & \frac{(c\sigma^2\sin\zeta)(a^2+c^2)\,\hat{i}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{3/2}} - \frac{(c\sin\zeta)\,\hat{i}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} \\
& - \frac{(c\sigma^2\cos\zeta)(a^2+c^2)\,\hat{j}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{3/2}} + \frac{(c\cos\zeta)\,\hat{j}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} \\
& + \frac{\sigma a\,(a^2+\sigma^2)^{-1/2}\,\hat{k}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} - \frac{\sigma a(a^2+\sigma^2)^{1/2}(a^2+c^2)\,\hat{k}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{3/2}}
\end{aligned} \tag{50}$$

normal rate

$$\bar{N}_v = \frac{(-c\sigma\cos\zeta)\,\hat{i} - (c\sigma\sin\zeta)\,\hat{j}}{[a^2(a^2+\sigma^2)+c^2\sigma^2]^{1/2}} \tag{51}$$

normal rate

ELLIPSOID

$$x = [\sigma \sin \zeta, -\sigma \cos \zeta, c(1 - \sigma^2/a^2)^{1/2} + z_1]$$ surface description (52)

$$x_u = [\sin \zeta, -\cos \zeta, (-c\sigma/a^2)(1 - \sigma^2/a^2)^{-1/2}]$$ tangent to surface (53)

$$x_v = [\sigma \cos \zeta, \sigma \sin \zeta, 0]$$ tangent to surface (54)

$$x_{uu} = [0, 0, (-c\sigma^2/a^4)(1 - \sigma^2/a^2)^{-3/2} - c/a^2 (1 - \sigma^2/a^2)^{-1/2}]$$ tangential rate (55)

$$x_{uv} = [\cos \zeta, \sin \zeta, 0]$$ tangential rate (56)

$$x_{vv} = [-\sigma \sin \zeta, \sigma \cos \zeta, 0]$$ tangential rate (57)

$$E = x_u \cdot x_u = 1 + \frac{\sigma^2 c^2}{a^2(a^2 - \sigma^2)} \tag{58}$$

$$F = x_u \cdot x_v = 0 \qquad (\perp \text{ systems}) \tag{59}$$

$$G = x_v \cdot x_v = \sigma^2 \tag{60}$$

standard forms of (I)

$$e = \frac{x_{uu}\, x_u\, x_v}{(EG - F^2)^{1/2}} = \frac{-ca^2}{(a^2 - \sigma^2)\,[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} \tag{61}$$

$$f = \frac{x_{uv}\, x_u\, x_v}{(EG - F^2)^{1/2}} = 0 \qquad (\perp \text{ systems}) \tag{62}$$

$$g = \frac{x_{vv}\, x_u\, x_v}{(EG - F^2)^{1/2}} = \frac{-c\sigma^2}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} \tag{63}$$

standard forms of (II)

$$k_1 = \frac{e}{E} = \frac{-ca^4}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{3/2}}$$ principal normal curvature (64)

$$k_2 = \frac{g}{G} = \frac{-c}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}}$$ principal normal curvature (65)

Rodrigues' formula for $(k_1) \rightarrow \bar{N}_u\, du = -k_1\, \bar{x}_u\, du$ (30)

$$(k_2) \rightarrow \bar{N}_v\, dv = -k_2\, \bar{x}_v\, dv \qquad (31)$$

$$\boxed{\bar{N} = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{c\sigma \sin\zeta\, \hat{i} - c\sigma\cos\zeta\, \hat{j} + a(a^2 - \sigma^2)^{1/2}\, \hat{k}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}}} \quad \text{surface normal} \qquad (66)$$

$$\begin{aligned}\bar{N}_u = {} & \frac{(c\sigma^2 \sin\zeta)(a^2 - c^2)\,\hat{i}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{3/2}} + \frac{(c\sin\zeta)\,\hat{i}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} \quad \text{normal rate} \\ & - \frac{(c\sigma^2 \cos\zeta)(a^2 - c^2)\,\hat{j}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{3/2}} - \frac{(c\cos\zeta)\,\hat{j}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} \\ & - \frac{\sigma a(a^2 - \sigma^2)^{-1/2}\,\hat{k}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} + \frac{\sigma a(a^2 - \sigma^2)^{1/2}(a^2 - c^2)\,\hat{k}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{3/2}} \end{aligned} \qquad (67)$$

$$\bar{N}_v = \frac{(c\sigma\cos\zeta)\,\hat{i} + (c\sigma\sin\zeta)\,\hat{j}}{[a^2(a^2 - \sigma^2) + c^2\sigma^2]^{1/2}} \quad \text{normal rate} \qquad (68)$$

SPHERE

$x = [\sigma\sin\zeta, -\sigma\cos\zeta, (c^2 - \sigma^2)^{1/2} + z_1]$	surface description	(69)
$\bar{x}_u = [\sin\zeta, -\cos\zeta, -c(c^2 - \sigma^2)^{-1/2}]$	tangent to surface	(70)
$\bar{x}_v = [\sigma\cos\zeta, \sigma\sin\zeta, 0]$	tangent to surface	(71)
$\bar{x}_{uu} = [0, 0, -\sigma^2(c^2 - \sigma^2)^{-3/2} - (c^2 - \sigma^2)^{-1/2}]$	tangential rate	(72)
$\bar{x}_{uv} = [\cos\zeta, \sin\zeta, 0]$	tangential rate	(73)
$\bar{x}_{vv} = [-\sigma\sin\zeta, \sigma\cos\zeta, 0]$	tangential rate	(74)

$$\left.\begin{aligned} & E = \bar{x}_u \cdot \bar{x}_u = 1 + \frac{\sigma^2}{(c^2 - \sigma^2)} && (75) \\ & F = \bar{x}_u \cdot \bar{x}_v = 0 \qquad (\perp \text{ systems}) && (76) \\ & G = \bar{x}_v \cdot \bar{x}_v = \sigma^2 && (77) \end{aligned}\right\} \quad \text{standard forms of (I)}$$

$$e = \frac{\bar{x}_{uu}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{-c}{(c^2 - \sigma^2)} \tag{78}$$

$$f = \frac{\bar{x}_{uv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = 0 \qquad (\perp \text{ systems}) \qquad \text{standard forms of (II)} \tag{79}$$

$$g = \frac{\bar{x}_{vv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{-\sigma^2}{c} \tag{80}$$

$$\boxed{k_1 = \frac{e}{E} = -\frac{1}{c}} \qquad \text{principal normal curvature} \tag{81}$$

$$\boxed{k_2 = \frac{g}{G} = -\frac{1}{c}} \qquad \text{principal normal curvature} \tag{82}$$

Rodrigues' formula for $(k_1) \rightarrow N_u\,du = -k_1\,\bar{x}_u\,du$ (30)

$(k_2) \rightarrow N_v\,dv = -k_2\,\bar{x}_v\,dv$ (31)

$$\boxed{N = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{\sigma \sin\zeta\,\hat{i} - \sigma\cos\zeta\,\hat{j} + (c^2 - \sigma^2)^{1/2}\,\hat{k}}{c}} \qquad \text{surface normal} \tag{83}$$

$$N_u = \frac{\sin\zeta\,\hat{i} - \cos\zeta\,\hat{j} - \sigma(c^2 - \sigma^2)^{-1/2}\,\hat{k}}{c} \qquad \text{normal rate} \tag{84}$$

$$N_v = \frac{\sigma\cos\zeta\,\hat{i} + \sigma\sin\zeta\,\hat{j}}{c} \qquad \text{normal rate} \tag{85}$$

CONE

$\bar{x} = (\sigma \sin \zeta, -\sigma \cos \zeta, c\sigma + z_1)$	surface description	(86)
$\bar{x}_u = (\sin \zeta, -\cos \zeta, c)$	tangent to surface	(87)
$\bar{x}_v = (\sigma \cos \zeta, \sigma \sin \zeta, 0)$	tangent to surface	(88)
$\bar{x}_{uu} = (0, 0, 0)$	tangential rate	(89)
$\bar{x}_{uv} = (\cos \zeta, \sin \zeta, 0)$	tangential rate	(90)
$\bar{x}_{vv} = (-\sigma \sin \zeta, \sigma \cos \zeta, 0)$	tangential rate	(91)

$$E = \bar{x}_u \cdot \bar{x}_u = 1 + c^2 \qquad (92)$$

$$F = \bar{x}_u \cdot \bar{x}_v = 0 \qquad (\perp \text{ systems}) \quad \text{standard forms of (I)} \qquad (93)$$

$$G = \bar{x}_v \cdot \bar{x}_v = \sigma^2 \qquad (94)$$

$$e = \frac{\bar{x}_{uu}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = 0 \qquad (95)$$

$$f = \frac{\bar{x}_{uv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = 0 \qquad (\perp \text{ systems}) \quad \text{standard forms of (II)} \qquad (96)$$

$$g = \frac{\bar{x}_{vv}\,\bar{x}_u\,\bar{x}_v}{(EG - F^2)^{1/2}} = \frac{\sigma c}{(1 + c^2)^{1/2}} \qquad (97)$$

$k_1 = \dfrac{e}{E} = 0$	principal normal curvature	(98)
$k_2 = \dfrac{g}{G} = \dfrac{c}{\sigma(1 + c^2)^{1/2}}$	principal normal curvature	(99)

Rodrigues' formula for $(k_1) \rightarrow \bar{N}_u \, du = -k_1 \, \bar{x}_u \, du$ (30)

$$(k_2) \rightarrow \bar{N}_v \, dv = -k_2 \, \bar{x}_v \, dv \tag{31}$$

$$\boxed{N = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{-c \sin \zeta \, \hat{i} + c \cos \zeta \, \hat{j} + \hat{k}}{(1 + c^2)^{1/2}}} \quad \text{surface normal} \tag{100}$$

$$N_u = 0 \quad \text{normal rate} \tag{101}$$

$$\bar{N}_v = \frac{-c \cos \zeta \, \hat{i} - c \sin \zeta \, \hat{j}}{(1 + c^2)^{1/2}} \quad \text{normal rate} \tag{102}$$

PARABOLIC TORUS

$$\bar{x} = \left[x, \left(\frac{x^2}{4F} + a\right) \sin \xi, \left(\frac{x^2}{4F} + a\right) \cos \xi + z_1\right] \quad \text{surface description} \tag{103}$$

$$\bar{x}_x = \left(1, \frac{x}{2F} \sin \xi, \frac{x}{2F} \cos \xi\right) \quad \text{tangent to surface} \tag{104}$$

$$\bar{x}_\xi = \left[0, \left(\frac{x^2}{4F} + a\right) \cos \xi, -\left(\frac{x^2}{4F} + a\right) \sin \xi\right] \quad \text{tangent to surface} \tag{105}$$

$$\bar{x}_{xx} = \left(0, \frac{\sin \xi}{2F}, \frac{\cos \xi}{2F}\right) \quad \text{tangential rate} \tag{106}$$

$$\bar{x}_{x\xi} = \left(0, \frac{x}{2F} \cos \xi, \frac{-x}{2F} \sin \xi\right) \quad \text{tangential rate} \tag{107}$$

$$\bar{x}_{\xi\xi} = \left[0, -\left(\frac{x^2}{4F} + a\right) \sin \xi, -\left(\frac{x^2}{4F} + a\right) \cos \xi\right] \quad \text{tangential rate} \tag{108}$$

$$\left.\begin{array}{ll} E = \bar{x}_x \cdot \bar{x}_x = 1 + \left(\dfrac{x}{2F}\right)^2 & \\ F = \bar{x}_x \cdot \bar{x}_\xi = 0 & (\perp \text{ systems}) \\ G = \bar{x}_\xi \cdot \bar{x}_\xi = \left(\dfrac{x^2}{4F} + a\right)^2 & \end{array}\right\} \text{standard forms of (I)} \qquad \begin{array}{r}(109)\\(110)\\(111)\end{array}$$

$$\left.\begin{array}{ll} e = \dfrac{\bar{x}_{xx}\,\bar{x}_x\,\bar{x}_\xi}{(EG - F^2)^{1/2}} = (4F^2 + x^2)^{-1/2} & \\ f = \dfrac{\bar{x}_{x\xi}\,\bar{x}_x\,\bar{x}_\xi}{(EG - F^2)^{1/2}} = 0 & (\perp \text{ systems}) \\ g = \dfrac{\bar{x}_{\xi\xi}\,\bar{x}_x\,\bar{x}_\xi}{(EG - F^2)^{1/2}} = \dfrac{-(4Fa + x^2)}{2(4F^2 + x^2)^{1/2}} & \end{array}\right\} \text{standard forms of (II)} \qquad \begin{array}{r}(112)\\(113)\\(114)\end{array}$$

$$\boxed{\begin{array}{l} k_1 = 4F^2(4F^2 + x^2)^{-3/2} \\ k_2 = -8F^2(4F^2 + x^2)^{-1/2}(x^2 + 4Fa)^{-1} \end{array}} \quad \begin{array}{l} \text{principal normal curvature} \quad (115) \\ \text{principal normal curvature} \quad (116) \end{array}$$

Rodrigues' formula for $(k_1) \rightarrow \bar{N}_u\, du = -k_1\, \bar{x}_u\, du$ (30)

$(k_2) \rightarrow \bar{N}_v\, dv = -k_2\, \bar{x}_v\, dv$ (31)

$$\boxed{\bar{N} = \frac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|} = \frac{-x\,\hat{i} + 2F \sin\xi\,\hat{j} + 2F\cos\xi\,\hat{k}}{(4F^2 + x^2)^{1/2}}} \quad \text{surface normal} \qquad (117)$$

$$\bar{N}_u = [x^2(4F^2 + x^2)^{-3/2} - (4F^2 + x^2)^{-1/2}]\,\hat{i} \qquad \text{normal rate}$$
$$- [2Fx \sin\xi\,(4F^2 + x^2)^{-3/2}]\,\hat{j} - [2Fx\cos\xi(4F^2 + x^2)^{-3/2}]\,\hat{k} \qquad (118)$$

$$\bar{N}_v = [2F\cos\xi(4F^2 + x^2)^{-1/2}]\,\hat{j} - [2F\sin\xi(4F^2 + x^2)^{-1/2}]\,\hat{k} \quad \text{normal rate} \qquad (119)$$

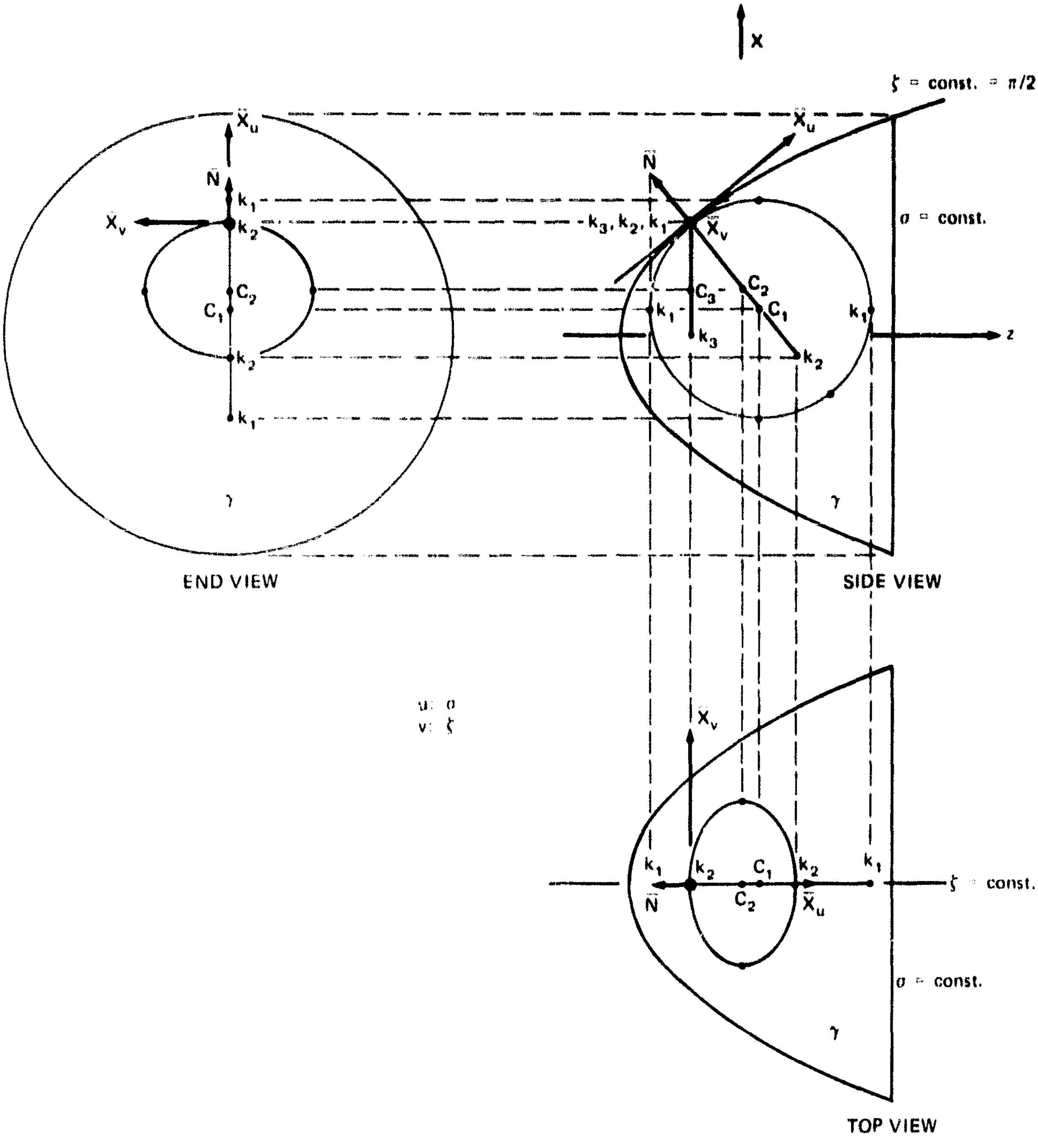

Figure 2. Principal Normal Curvatures of the (σ, ζ) Parameterization

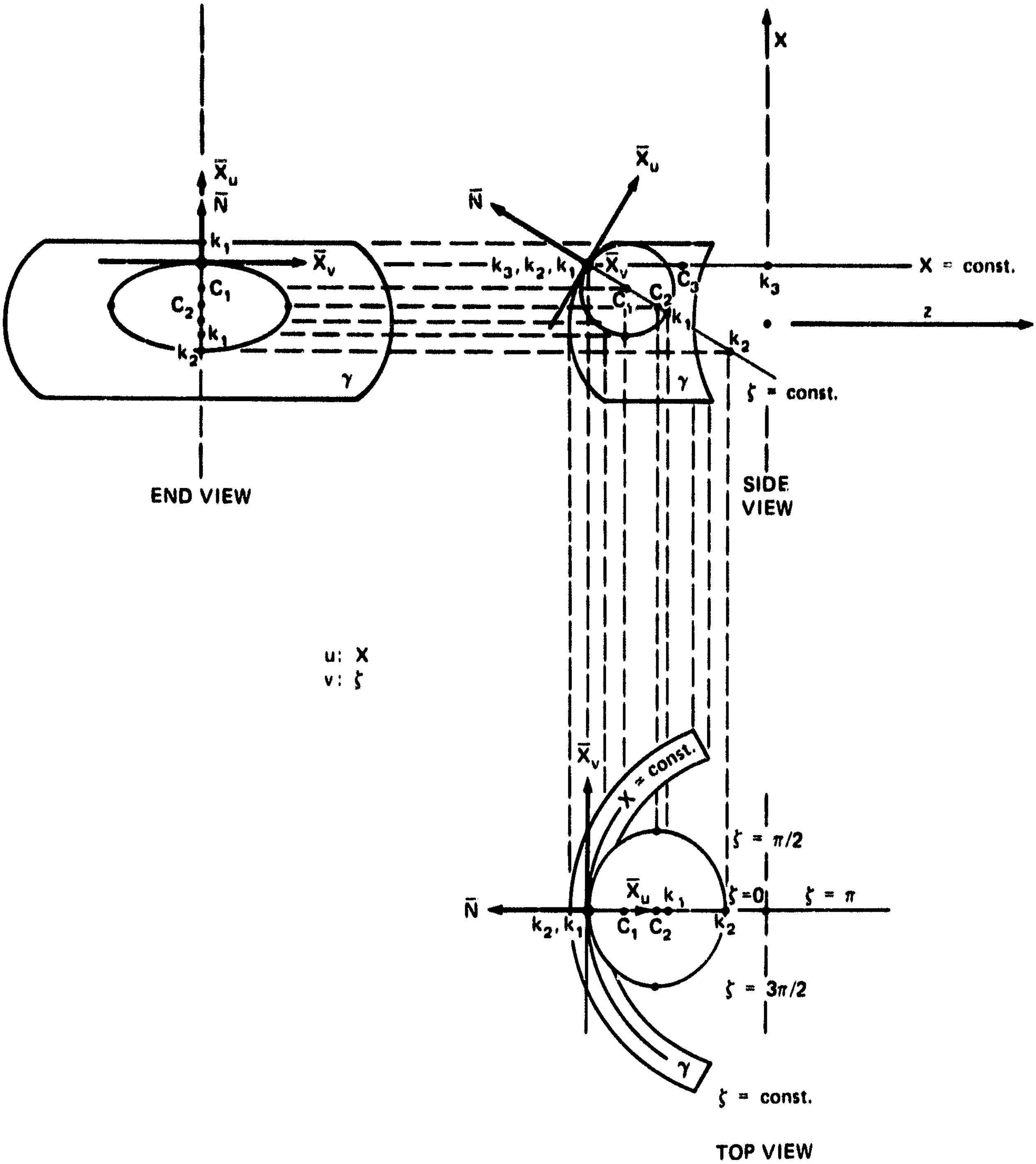

Figure 3. Principal Normal Curvatures of the (x, ξ) Parameterization

GRAPHIC PRESENTATION

Two illustrations are now presented. Figures (2) and (3) are representative of the (σ, ζ) and (x, ξ) parameterizations, respectively, and depict the principal normal curvatures (k_1) and (k_2) as well as their centers of curvature (c_1) and (c_2). The surface normals are shown, together with the parametric lines (σ = const., ζ = const., x = const., ξ = constant), and the tangent lines $(\bar{x}_u, \bar{x}_v)$. The circles of curvature (k_3) and their centers (c_3) are associated with the generating arc, and are seen to be distinct, in general, from (k_1) and (k_2), and (c_1) and (c_2).

As a plane containing the normal $(\bar{N})$ is rotated about the normal, the center of curvature varies between the extrema (c_1) and (c_2). This definite motion along the normal produces a description of the surface curvature at the point being considered. Ref. 8, p. 184. Appropriate plots of (c) versus angle (α) may be constructed as graphic aids. The curvature of every normal section is completely determined by the principal curvatures and the angle which the normal section makes with the principal directions according to Euler's theorem.

EDGE DIFFRACTION

The principal normal curvatures (k_1) and (k_2) which have been shown to lead to generic normal curvature via Euler's theorem have a wide range of application in diffraction theory. Ref. 2, p. 67. Curvature matrices are used extensively; one-half of the trace or spur equals the mean curvature (k_m), and the determinant equals the Gaussian curvature (k_g).

$$\bar{\bar{Q}} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{120}$$

For a local Cartesian coordinate frame at a point (0) on a surface γ, a typical point in the neighborhood of (0) may be represented by

$$z = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \bar{\bar{Q}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0(x_{1,2}^3). \tag{121}$$

When basis vectors $(\bar{x}_1, \bar{x}_2)$ coincide with the principal directions $(\bar{a}_1, \bar{a}_2)$ the curvature matrix $\bar{\bar{Q}}$ is diagonal as in equation (120), above. If $(\bar{x}_1, \bar{x}_2)$ and $(\bar{a}_1, \bar{a}_2)$ are skew in the amount (ψ),

$$\bar{\bar{Q}} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}^T \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}. \tag{122}$$

Other adaptations can be found in the literature. A family of parallel wavefront surfaces (δ = const.) is considered in terms of a local Cartesian frame as before. Assuming $(\bar{x}_1, \bar{x}_2)$ are the principal directions there exists a coordinate system (x_1, x_2, δ), such that the curvature matrix of $\omega(\delta)$ takes the form

$$\bar{\bar{Q}}(\delta) = \begin{bmatrix} \dfrac{1}{R_1 + \delta} & 0 \\ 0 & \dfrac{1}{R_2 + \delta} \end{bmatrix} \tag{123}$$

When the basis vectors do not coincide with the principal directions, the amendment of equation (122) applies. Here,

$$k_1 = \frac{1}{R_1}, \quad k_2 = \frac{1}{R_2}, \tag{124}$$

where (R_1) and (R_2) are the principal radial of curvature.

In the calculation of ray amplitudes

$$\left[\frac{\det \bar{\bar{Q}}(\delta)}{\det \bar{\bar{Q}}(\delta_0)}\right]^{1/2} = \frac{(R_1 + \delta_0)^{1/2}(R_2 + \delta_0)^{1/2}}{(R_1 + \delta)^{1/2}(R_2 + \delta)^{1/2}} \tag{125}$$

for two points (δ_0) and (δ) on a given ray.

Additional information pertaining to the theory of diffraction and curvature matrices may be found in Ref. 9, p. 94-114, and Ref. 10, p. 1-33. A connection between the curvature matrix and the Weingarten equations may be found in Ref. 11, p. 67. These topics lie beyond the scope of the present document.

CONCLUSION

The principal normal curvatures of six surfaces of revolution frequently encountered in microwave antenna analysis were developed via orthogonal coordinate nets. No particular significance was attached to the sign (±) of (k_1) and (k_2), but the change in sign is considered significant. An arbitrary convention may be introduced. It was shown that the results satisfied Rodrigues' formula.

Parameterizations of virtually any surface may be dealt with by the means illustrated in this document to obtain normal curvature, area, etc.

The tangents along lines of curvature and the normal to surface (γ) comprise an orthogonal triad ($\bar{x}_u$, $\bar{x}_v$, $\bar{N}$) that is distinct, in general, from the orthogonal triad ($\bar{t}$, $\bar{n}$, $\bar{b}$) of Serret (1851) and Frenet (1847), which pertains to curves (c) lying on surface (γ). The distinction is brought out more strongly by the following. In the latter case, the derivatives ($\bar{t}'$, $\bar{n}'$, $\bar{b}'$) are given by

$$\bar{t}' = \bar{D} \times \bar{t}, \bar{n}' = \bar{D} \times \bar{n}, \bar{b}' = \bar{D} \times \bar{b} \tag{126}$$

where the $\bar{D}$ is the Darboux vector

$$\bar{D} = \tau \bar{t} + \kappa \bar{b}. \tag{127}$$

In the former case, the continuous derivatives

$$\bar{x}_{uu}, \bar{x}_{uv}, \bar{x}_{vv}, \bar{N}_u \text{ and } \bar{N}_v$$

are expressed as linear combinations of $\bar{x}_u$, $\bar{x}_v$, $\bar{N}$ via the Gauss-Weingarten equations in an analogous manner:

$$\left.\begin{aligned} \bar{x}_{uu} &= \Gamma_{11}^{1} \bar{x}_u + \Gamma_{11}^{2} \bar{x}_v + e \bar{N} & (128)\\ \bar{x}_{uv} &= \Gamma_{12}^{1} \bar{x}_u + \Gamma_{12}^{2} \bar{x}_v + f \bar{N} & (129)\\ \bar{x}_{vv} &= \Gamma_{22}^{1} \bar{x}_u + \Gamma_{22}^{2} \bar{x}_v + g \bar{N} & (130) \end{aligned}\right\} \text{Gauss}$$

$$\left.\begin{aligned} \bar{N}_u &= \beta_1^1 \bar{x}_u + \beta_1^2 \bar{x}_v & (131)\\ \bar{N}_v &= \beta_2^1 \bar{x}_u + \beta_2^2 \bar{x}_v & (132) \end{aligned}\right\} \text{Weingarten}$$

Ref. 6, p. 202, Ref. 4, p. 22.

The difference between the normal curvature of a surface (k_0 in Ref. 12, p. 118) and the curvature of a curve (k in Ref. 12, p. 117) lying on the surface is also illustrated by the relationship.

$$k_0 = k \cos \vartheta = \frac{II}{I}, \tag{133}$$

where (ϑ) is the angle between the principal normal to the curve and the normal to the surface. The direction du : dv for k_0 is the direction of the tangent for the curve.

ACKNOWLEDGMENTS

The author acknowledges input from the cited references, and helpful discussions with Mr. Jack Healy of Sigma Data Services Corporation, under contract to NASA/Goddard Space Flight Center.

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APPENDIX A

Cylindrical Parameterization (Summary)

$x = \sigma \sin \zeta \qquad y = -\sigma \cos \zeta$ (1-A)

paraboloids $\quad z = \sigma^2/4F + z_1$ (2-A)

hyperboloids $\quad z = c(1 + \sigma^2/a^2)^{1/2} + z_1$ (3-A)

ellipsoids $\quad z = c(1 - \sigma^2/a^2)^{1/2} + z_1$ (4-A)

spheres $\quad z = (c^2 - \sigma^2)^{1/2} + z_1$ (5-A)

cones $\quad z = c\sigma + z_1$ (6-A)

Tangents to Surfaces

$\bar{\rho}_\sigma = \partial\bar{\rho}/\partial\sigma, \quad \bar{\rho}_\zeta = \partial\bar{\rho}/\partial\zeta, \quad \bar{\rho} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z = (x, y, z)$ (7-A)

$\bar{\rho}_\zeta = (\sigma \cos \zeta, \quad \sigma \sin \zeta, 0)$ for rotationally symmetric surfaces.

paraboloids $\quad \bar{\rho}_\sigma = (\sin \zeta, -\cos \zeta, \sigma/2F)$ (8-A)

hyperboloids $\quad \bar{\rho}_\sigma = [\sin \zeta, -\cos \zeta, \sigma c/a(a^2 + c^2)^{1/2}]$ (9-A)

ellipsoids $\quad \bar{\rho}_\sigma = [\sin \zeta, -\cos \zeta, -\sigma c/a(a^2 - c^2)^{1/2}]$ (10-A)

spheres $\quad \bar{\rho}_\sigma = [\sin \zeta, -\cos \zeta, -\sigma/(c^2 - \sigma^2)^{1/2}]$ (11-A)

cones $\quad \bar{\rho}_\sigma = (\sin \zeta, -\cos \zeta, c]$ (12-A)

Normals to Surfaces

$$\bar{n} = (\bar{\rho}_\sigma \times \bar{\rho}_\zeta)/|\bar{\rho}_\sigma \times \bar{\rho}_\zeta| = (\bar{\rho}_\sigma \times \bar{\rho}_\zeta)/(EG - F^2)^{1/2} \qquad (13\text{-A})$$

Differential Areas

$$dS = (EG - F^2)^{1/2}\, d\sigma\, d\zeta \qquad (14\text{-A})$$

$$E = x_\sigma^2 + y_\sigma^2 + z_\sigma^2 \qquad (15\text{-A})$$

$$F = x_\sigma x_\zeta + y_\sigma y_\zeta + z_\sigma z_\zeta \quad (16\text{-A})$$

$$G = x_\zeta^2 + y_\zeta^2 + z_\zeta^2 \quad (17\text{-A})$$

paraboloids $\quad dS = \sigma(1 + \sigma^2/4F^2)^{1/2}\, d\sigma\, d\zeta \quad (18\text{-A})$

hyperboloids $\quad dS = \sigma[1 + \sigma^2 c^2/a^2 (a^2 + \sigma^2)]^{1/2}\, d\sigma\, d\zeta \quad (19\text{-A})$

ellipsoids $\quad dS = \sigma[1 + \sigma^2 c^2/a^2 (a^2 - \sigma^2)]^{1/2}\, d\sigma\, d\zeta \quad (20\text{-A})$

spheres $\quad dS = \sigma c/(c^2 - \sigma^2)^{1/2}\, d\sigma\, d\zeta \quad (21\text{-A})$

cones $\quad dS = \sigma(c^2 + 1)^{1/2}\, d\sigma\, d\zeta \quad (22\text{-A})$